Classification of purely infinite graph algebra with finitely many ideals

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Ultimate Goal

Problem

Find a complete algebraic invaraiant for
{ separable, nuclear, purely infinite, stable
C*-algebras with finitely many ideals }

No non-trivial ideal: K_0 and K_1 (Kirchberg, Phillips)

One non-trivial ideal: 6-term exact sequence
(Rørdam)

How about Cuntz-Krieger algebras?
graph algebras? real rank zero C*-alg?

References

[EKTW] Søren Eilers, Takeshi Katsura,
Mark Tomforde, and James West,
"*The ranges of K-theoretic invariants for non-simple graph algebras*", preprint 2011.

[ABK] Sara Arklint, Rasmus Bentmann and Takeshi Katsura,
"*Reduction of filtered K-theory and a characterization of Cuntz-Krieger algebras*",
in preparation.

based on papers [Restorff], [Kirchberg],
[Meyer-Nest], [Bentmann-Köhler], , , ,

Purely infinite C*-algebras

Definition

A C*-algebra A is purely infinite

$\Longleftrightarrow$ every non-zero positive element in A is properly infinite

Fact

A: simple separable nuclear C-algebra*

A is purely infinite $\Longleftrightarrow$ $A \cong A \otimes \mathcal{O}_\infty$

The same is true for separable nuclear C*-alg. with finitely many ideals (by Fact).

In this talk "purely infinite" sometimes means the strictly stronger condition $A \cong A \otimes \mathcal{O}_\infty$.

simple purely infinite C*-algebras

- ('77) Cuntz introduced Cuntz algebras $\mathcal{O}_n$.
- ('78) Pimsner-Popa classified $\mathcal{O}_n$ up to isom by *Ext*
- ('80) Cuntz-Krieger introduced Cuntz-Krieger algebra $\mathcal{O}_A$, and showed stable isom of $\mathcal{O}_A$ from flow equiv of SFT X_A
- ('84) Franks classified irreducible SFT X_A up to flow equiv by signed Bowen-Franks group
- ('95) Rørdam classified simple Cuntz-Krieger algebras $\mathcal{O}_A$ up to (stable) isom by K_0-groups
- ('95) Elliott-Rørdam classified the “classifiable class” of simple purely infinite C*-algebras up to isom by K-theory

Kirchberg algebras and UCT

Definition

Kirchberg algebra = simple, separable,
purely infinite, nuclear C*-algebra

Theorem (Rosenberg-Schochet '87)

C-algebra A in Bootstrap class*

$\Longleftrightarrow$ *A satisfies UCT for KK*

$\Longleftrightarrow$ *KK-equivalence for A*

= isomorphism of K-groups for A

Kirchberg-Phillips classification

Theorem (Kirchberg, Phillips '00)

Kirchberg algebras in Bootstrap class are classified up to stable isomorphism by K_0-groups and K_1-groups.

and up to isomorphism by K_0-groups and K_1-groups and the position of the unit.

range of invariants:
all pairs of countable abelian groups
(Elliott-Rørdam '95)

non-simple purely infinite C*-algebras

- ('80) Cuntz-Krieger showed stable isom of $\mathcal{O}_A$ from flow equiv of SFT X_A
- ('95) Huang classified some of $\mathcal{O}_A$ up to isom by filtered Bowen-Franks group
- ('97) Rørdam classified purely infinite C*-algebras with one ideal up to stable isom by 6-term sequence of K-groups
- ('03) Boyle-Huang introduced K-web
- ('06) Restorff classified $\mathcal{O}_A$ up to stable isom by filtered K-theory
- ('08) Eilers-Restorff^2-Ruiz classified purely infinite C*-algebras with one ideal up to isom by 6-term sequence of K-groups with "unit"

C*-algebras over topological spaces

X: topological space (finite, T_0)

Definition (Meyer-Nest '08)

C*-algebra over X = C*-algebra A
& continuous map $\psi\colon \operatorname{Prim}(A) \to X$
Such A is tight if ψ is a homeomorphism

$\operatorname{Prim}(A) =$ the primitive ideal space of A

$\mathbb{LC}(X) := \{$locally closed subsets of $X\}$
$U \in \mathbb{LC}(X)$ open $\rightsquigarrow A(U) \triangleleft A$
$Y \in \mathbb{LC}(X) \rightsquigarrow Y = U \setminus V$ for open $V \subset U$
$\rightsquigarrow A(Y) := A(U)/A(V)$
($A(Y)$ does not depend on the choices of U, V)

K-web (=filtrated (filtered) K-theory)

$Y \in \mathbb{LC}(X)$ and $Z \subset Y$ open $\rightsquigarrow Z, Y \setminus Z \in \mathbb{LC}(X)$

$$\rightsquigarrow \; 0 \longrightarrow A(Z) \longrightarrow A(Y) \longrightarrow A(Y \setminus Z) \longrightarrow 0$$

$$\rightsquigarrow \quad \begin{array}{ccccc} K_0\big(A(Z)\big) & \xrightarrow{i} & K_0\big(A(Y)\big) & \xrightarrow{r} & K_0\big(A(Y \setminus Z)\big) \\ \delta \uparrow & & & & \downarrow \delta \\ K_1\big(A(Y \setminus Z)\big) & \xleftarrow[r]{} & K_1\big(A(Y)\big) & \xleftarrow[i]{} & K_1\big(A(Z)\big) \end{array}$$

concrete and abstract K-webs

X: topological space (finite, T_0)

Definition

A: C*-algebra over X

$$K_X(A) := \Big(K_*(A(Y))_{Y\in\mathbb{LC}(X)}, (i, r, \delta)\Big)$$

concrete K-web

Meyer-Nest considered categories and natural transformations to get an abstract K-web:

$$K_X^{\mathrm{MN}}(A) = \Big(K_*(A(Y))_{Y\in\mathbb{LC}(X)}, (\text{“natural maps”})\Big)$$

Problem

$$K_X^{\mathrm{MN}}(A) \overset{?}{=} K_X(A)$$

Kirchberg X-algebra

X: topological space

Definition

Kirchberg X-algebras = tight, separable, nuclear, purely infinite C*-algebras over X

Theorem (Kirchberg '00)

A, B: Kirchberg X-algebras

A and B are stably isomorphic over X

$\Longleftrightarrow$ A and B are $KK(X)$-equivalent

Universal Coefficient Theorem

Definition (Meyer-Nest '09)

C*-algebra A over X satisfies UCT for X
"$\Longleftrightarrow$" $KK(X)$-equivalence for A
= isomorphism of abs. K-web $K_X^{\mathrm{MN}}(A)$

Theorem (Meyer-Nest '09, Bentmann-Köhler '11)

For a finite T_0 space X, TFAE:

1. *A in Bootstrap class for $X \Rightarrow A$: UCT for X*
2. *The class of stable Kirchberg X-algebras in Bootstrap class is classified by $K_X^{MN}(-)$*
3. *X is a disjoint union of "accordion spaces"*

finite T_0 space

X: a finite set

$$\{T_0\text{-topologies on } X\} \overset{1:1}{\leftrightarrow} \{\text{partial orders on } X\}$$

$$\overline{\{x\}} \subset \overline{\{y\}} \iff x \leq y$$

a partial order on X can be visualized
by drawing arrow from y to x
if $x < y$ and no z satisfies $x < z < y$

remarks on [MN] and [BK]

Theorem (Meyer-Nest '09, Bentmann-Köhler '11)

For a finite T_0 space X, TFAE:

1. *The class of stable Kirchberg X-algebras in Bootstrap class is classified by $K_X^{MN}(-)$*
2. *X is a disjoint union of "accordion spaces"*

$K_X^{\mathrm{MN}}(A) = K_X(A)$ for accordion space X

(Bentmann-Köhler '11)

For X not a disjoint union of accordion space, $K_X^{\mathrm{MN}}(-)$ (and $K_X(-)$) is not a complete invariant.

Problem

Find invariants ($\supset K_X^{\mathrm{MN}}(-)$) and show UCT.

Restorff's theorem

Theorem (Restorff)

The class of Cuntz-Krieger algebras $\mathcal{O}_A$ are classified up to stable isom by a part of K-web.

Proof uses results on dynamical systems.

Problem

Give C*-algebraic proof.

How about purely infinite graph algebras?

How about more general purely infinite C*-alg?

Graph algebras

Definition

$E = (E^0, E^1, s, r)$: (directed) graph

$\iff$ E^0, E^1: countable sets

$s, r : E^1 \to E^0$

Definition

A graph algebra $C^*(E)$ is generated

by pairwise $\perp$ projections $\{p_v\}_{v \in E^0}$ and

partial isometries $\{s_e\}_{e \in E^1}$ with $\perp$ ranges s.t.

1. $s_e^* s_e = p_{s(e)}, \quad s_e s_e^* \leq p_{r(e)}$
2. $p_v = \sum_{e \in r^{-1}(v)} s_e s_e^*$ if $0 < |r^{-1}(v)| < \infty$

K-web of graph algebras

E: graph with Condition (K)

Set $A := C^*(E)$ and $X := \mathrm{Prim}(A)$.

$\exists$ description of X and concrete K-web $K_X(A)$
in terms of graph E

$K_X(A)$ satisfies

1. all $K_1(A(Y))$ is free for all $Y \in \mathbb{LC}(X)$,
2. $\delta : K_0(A(Y \setminus Z)) \to K_1(A(Z))$ is zero
   for all $Y \in \mathbb{LC}(X)$ and $Z \subset Y$ open.

Definition

We say $K_X(A)$ is graph-like
if it satisfies the two conditions above.

Classification of purely infinite graph alg.

Theorem (Arklint-Bentmann-Katsura)

X: accordion space
Every Kirchberg X-algebras with graph-like K-web is stably isomorphic to graph algebras.

Theorem (ABK)

X: accordion space
A C-algebra A is isomorphic to a Cuntz-Krieger algebra whose primitive ideal space is X*
$\iff$ A is a unital Kirchberg X-algebra with Cuntz-Krieger-like K-web.

Both Thms may hold for all finite T_0 space X

Proof of main theorem of [ABK]

X: accordion space

Theorem (Arklint-Bentmann-Katsura)

Every Kirchberg X-algebras with graph-like K-web is stably isomorphic to graph algebras.

This follows from

Theorem (K + MN + BK)

Kirchberg X-algebras are classified up to stable isom by $K_X(-)$

Theorem (ABK)

If $K_X(A)$ is graph-like, then $\exists$ graph E s.t. $C^*(E)$ is a Kirchberg X-algebra with $K_X(C^*(E)) \cong K_X(A)$

Construction of graph in [ABK]

X: accordion space
(or more generally BDP space)

Theorem (ABK)

If $K_X(A)$ is graph-like, then $\exists$ graph E s.t. $C^*(E)$ is a Kirchberg X-algebra with $K_X(C^*(E)) \cong K_X(A)$

This follows from

Proposition (ABK)

If $K_X(A)$ is graph-like, then $K_X(A)$ is recovered from a part of $K_X(A)$ (as in Restorff's result).

and the main theorem of [EKTW]

Main theorem of [EKTW]

Theorem (Eilers-Katsura-Tomforde-West)

For an exact sequence $\mathcal{E}$:

$$\begin{array}{ccccc} G_1 & \longrightarrow & G_2 & \longrightarrow & G_3 \\ \uparrow & & & & \downarrow 0 \\ F_3 & \longleftarrow & F_2 & \longleftarrow & F_1 \end{array}$$

with free F_i,

$\exists$ *graph algebra A with an ideal I*

s.t. 6-term sequence of K-groups for $I \subset A$

is isomorphic to $\mathcal{E}$

We can control the graphs for I, A/I

as well as the position of unit (if it exists).